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A general theory describing the time dependent mechanical breakdown phenomena for homogeneous, oriented solids subjected to time dependent stresses is formulated. The kinetic nature of the microscopic molecular behavior is taken into account. Approximate solutions for time required to fracture under repeated loading conditions are examined. Some parameters involved in the theory are discussed. Within a large range of various applied periodic tensile stresses the logarithm of time-to-fracture is found to be almost linearly related with the maximum amplitude of the applied stress. As this maximum amplitude gets smaller and smaller the time required for fracture becomes greater and greater and approaches to infinity for a certain small limiting value.

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INTRODUCTION

This report is a continuation of a series of articles[1-4] on the development of a theory concerning the time dependent nature of the mechanical breakdown of homogeneous, oriented, and stressed solids. We shall consider here the strength behavior of a solid affected by microscopic changes when subjected to time dependent loads, particularly those represented by periodic loading functions.

The theory of breaking kinetics is concerned with the problem of calculating the expected distribution of times-to-break for a representative element oriented in different possible directions and in turn for an entire solid composed of a system of elements when subjected to time dependent loading histories. During the loading process, until the occurrence of fracture, the variation in the molecular forces represented by the elements throughout the solid must be considered in the analysis. The strength behavior such as the ultimate strength of the solid is intimately tied to the strength distribution and orientation of all the individual microscopic elements in the entire solid. In order to take into account these features, the mathematical model used throughout the course of this analysis is a matrix of oriented elements embedded in an arbitrary domain. For simplicity, consider that the model consists of a large number of identical linear elements distributed continuously in the entire system. For such a system, the state of stress in the vicinity of a point in the solid may be expressed by the time dependent stress tensor σ_{ij} $(i,j = 1,2,3)$ in a rectangular coordinate system[1]:

$$\sigma_{ij}(\varepsilon,t) = \int \rho(\theta,\phi,\varepsilon)\,\psi(\theta,\phi,t)\,f(\theta,\phi,t)s_i s_j d\omega \tag{1}$$

where ε represents a state of finite homogeneous strain which serves as a measure of the orientation of the elements resulted from deformation, t identifies the time. $\rho(\theta,\phi,\varepsilon)$ is the probability density of the distribution function of orientation designated by spherical coordinates (θ,ϕ), $\psi(\theta,\phi,t)$ is the time dependent axial stress on any representative element along the direction defined by the spherical coordinates (θ,ϕ), $f(\theta,\phi,t)$ is the fraction of unbroken elements as a function of orientation and time, s_i and s_j are unit vectors, and $d\omega$ is an infinitesimal solid angle within which the elements are considered to be parallel. Integration is performed so as to cover all the possible orientation of the elements contained in the entire solid.

Depending upon the nature of molecular constitution, two limiting cases or their combinations may be realized. For a system of randomly oriented elements, if ε is a measure of homogeneous finite strain, then[1]

$$\rho(\theta,\phi,\varepsilon) = \frac{(1+\varepsilon)^3}{[\cos^2\theta + (1+\varepsilon)^3\sin^2\theta]^{3/2}}\,\rho(0) \tag{2}$$

where $\rho(0) = 1/4\pi$, a constant, represents the random distribution function of orientation. In the case of a system with randomly oriented and flexibly connected elements at their end joints[2]:

$$\rho(\theta,\phi,\varepsilon) = \frac{\alpha}{[\cos^2 \frac{\theta}{2} + \alpha \sin^2 \frac{\theta}{2}]^2}\, \rho(0) \tag{3}$$

where α is associated with ε as follows

$$\varepsilon = \frac{2}{\alpha-1}[\alpha + 1 + \frac{4\alpha}{\alpha-1} \ln \frac{\alpha+1}{2\alpha}] \tag{4}$$

The quantity $f(\theta,\phi,t)$ is introduced through considering the absolute reaction rate and nonlinear viscosity. It's time rate of change can be given as follows:

$$\frac{df(\theta,\phi,t)}{dt} = K_r[1 - f(\theta,\phi,t)] - K_b f(\theta,\phi,t) \tag{5}$$

where

$$K_r = \omega_r \exp[-\frac{U}{RT} - \gamma\psi(\theta,\phi,t)]$$

and (6)

$$K_b = \omega_b \exp[-\frac{U}{RT} + \beta\psi(\theta,\phi,t)]$$

are the rate coefficients for reformation and breakage of the elements respectively. ω_r, ω_b, γ, β, U and R are material constants and T is the absolute temperature.

The solution to the governing equations (1), (5) and either (2) or (3) together with an appropriate fracture criterion will enable us to predict the time required to fracture t_b for a solid.

Let us now consider the strength of a solid with its constituent elements arbitrarily oriented to any degree under a general state of stress. In the vicinity of a point under consideration, let the stress tensor be given in the following form:

$$\sigma_{ij}(t) = \sigma_{ij}^{o}\zeta(t) \tag{7}$$

σ_{ij}^{o} are constants and where $\zeta(t)$ is a piecewise continuous function. Substituting into (1) we obtain

$$\sigma_{ij}^{o}\zeta(t) = \int \rho(\theta,\phi,\varepsilon)\ f(\theta,\phi,t)\ \psi(\theta,\phi,t) s_i s_j d\omega \tag{8}$$

It can be easily shown from the above equation that we must have the following relation:

$$f(\theta,\phi,t)\ \psi(\theta,\phi,t) = \mu(\theta,\phi)\zeta(t) \tag{9}$$

where $\mu(\theta,\phi)$, an arbitrary function dependent solely upon orientation, is to be determined later. Equation (9) holds for every representative element in an arbitrary direction (θ,ϕ). Eliminating ψ from (5) and (9) we obtain,

$$\frac{df(\theta,\phi,t)}{dt} = -F[f(\theta,\phi,t),\mu(\theta,\phi)\zeta(t)] \tag{10}$$

where F is defined as

$$F[f,\mu\zeta] \equiv \Omega_b f \exp(\frac{\beta\mu\zeta}{f}) - \Omega_r(1 - f) \exp(- \frac{\gamma\mu\zeta}{f}) \tag{11}$$

and f is understood to be a function of orientation (θ,ϕ) and time t, μ, a function of (θ,ϕ) only and ζ, a function of t only.

Also

$$\Omega_b = \omega_b \exp(-\frac{U}{RT})$$

$$\Omega_r = \omega_r \exp(-\frac{U}{RT})$$

If we eliminate f instead of ψ from (5) and (9), then

$$\frac{d\psi(\theta,\phi,t)}{dt} = \Psi[\psi(\theta,\phi,t),\mu(\theta,\phi)\zeta(t)] \tag{12}$$

where Ψ is defined as

$$\Psi[\psi,\mu\zeta] \equiv \frac{\psi}{\zeta}\frac{d\zeta}{dt} + \Omega_b\psi \exp(\beta\psi) + \Omega_r\psi(\frac{\psi}{\mu\zeta} - 1) \exp(-\gamma\psi) \tag{13}$$

and ψ is understood to be a function of (θ,ϕ) and t. Equation (10) or (12) depends upon orientation of the representative element and will have different solutions along different directions. Assuming that all elements will break when either $f \to 0$ or $\psi \to \psi_b$ (say) where ψ_b is a constant, independent of orientation, that is fracture will occur when

$$t = t_b(\theta,\phi), \quad f(\theta,\phi,t) \to 0; \quad \text{or} \quad \psi(\theta,\phi,t) = \psi_b \tag{14}$$

depending upon whether (10) or (12) is considered. Also, for convenience without losing generality, we assume that all the elements are unbroken initially, and we have

$$f(\theta,\phi,0) = 1; \quad \psi(\theta,\phi,0) = \psi_o(\theta,\phi); \quad (t = 0) \tag{15}$$

Now our problem reduces to, firstly, finding the times-to-break $t_b(\theta,\phi)$ for individual elements from (10) or (12) with the help of (14) and (15); and secondly correlate the statistical time-to-break $\overline{t}_b$ for the entire solid with various values of $t_b(\theta,\phi)$. Let us assume that $\sigma_{ij}(t)$ in (7) are such that each representative element is under tension. Then focussing our attention on one representative element, the problem so defined by (10) or (12), together with (14) and (15) is well posed and has a solution. Also it can be shown that f and ψ are monotone decreasing and increasing functions respectively of time throughout the range $0 \leq t \leq t_b(\theta,\sigma)$. Moreover Ψ is real, finite and non-negative whereas F is real, finite and non-positive. If $\mu\zeta$ is denoted by σ and is a piecewise continuous function of t, then it follows from (9) that either f or ψ or both must be piecewise continuous. However, continuity of f is necessary because in a physical system as defined the number of unbroken elements would not change abruptly during a discontinuous variation in loading. A discontinuity in ψ arises because of (9) and is expected in physical systems even though f is continuous. Therefore, in the case of a given continuous loading function σ, either (10) or (12) together with (14) and (15) determine the solution of the problem completely. For a piecewise loading function σ however, we must solve (10) rather than (12). Now let us assume that

$$\sigma_{max}(\theta,\phi) \geq \sigma \geq \sigma_{min}(\theta,\phi) \qquad [0 \leq t \leq t_b(\theta,\phi)]$$

Then clearly the bounds on $t_b(\theta,\phi)$ are

$$\int_0^1 \frac{df}{F[f,\sigma_{min}]} \geq t_b(\theta,\phi) \geq \int_0^1 \frac{df}{F[f,\sigma_{max}]} \tag{16}$$

and equivalently in terms of ψ

$$\int_{\sigma_{min}}^{\psi_b} \frac{d\psi}{\Psi[\psi,\sigma_{min}]} \geq t_b(\theta,\phi) \geq \int_{\sigma_{max}}^{\psi_b} \frac{d\psi}{\Psi[\psi,\sigma_{max}]} \tag{17}$$

provided that the integrals exist. It may be mentioned at this stage that in order to get a finite upper bound, σ_{min} must be greater than some critical value $\sigma_{cr} > 0$ for which the upper bound becomes infinite[2,4]. Otherwise the results cease to be meaningful. For a constant applied load, i.e. σ = constant, the solution to either (10) or (12) can be easily obtained through integration by quadrature[2,3,4]. Also if the time dependent loading σ is such that its minimum value is of large magnitude, the influence of reformation processes on fracture becomes negligible and (12) reduces to

$$\frac{d\psi}{dt} = \frac{\psi}{\sigma}\frac{d\sigma}{dt} + \Omega_b\psi \exp(\beta\psi) \tag{18}$$

which yields a solution by quadrature provided that $\sigma = \sigma_o(\theta,\phi)\exp(ct)$ where σ_o depends upon (θ,ϕ) and c is a constant. Then the time-to-break $t_b(\theta,\phi)$ becomes

$$t_b(\theta,\phi) = \int_{\sigma_o(\theta,\phi)}^{\psi_b} \frac{d\psi}{\psi[c + \Omega_b \exp(\beta\psi)]} \tag{19}$$

which can be evaluated easily. We can obtain the same result as in (19) by considering equation (10) rather than (12). This is not surprising since equations (10) and (12) are derived from the same system of equations (5) and (9) and are completely equivalent. Such a simple result cannot be expected, however, if σ is not very large in order to warrant the use of (18) or is a periodic function involving low stress values. For low stresses, the reformation processes of the elements will influence the time dependent fracture considerably and cannot be neglected.

With regard to the calculation of $t_b(\theta,\phi)$ for cases in which the applied stresses and the resulting stresses $\sigma_{ij}(t)$ in the vicinity of a point under consideration oscillate with a period $\tau = 1/p$, it is recognized that the load σ on any representative element will also be periodic, thus

$$\sigma(\theta,\phi,t) = \sigma(\theta,\phi,t - n\tau) \tag{20}$$

where p is the frequency and n is the number of cycles elapsed. We restrict our discussion to all the σ, for all $\tau > 0$ that are piecewise continuous and

$$\min_{0 \leq t \leq t_b} \sigma(\theta,\phi,t) > \sigma_{cr}$$

Also the frequency p is assumed large and in turn is the number of cycles to fracture n_b. It does not seem possible to obtain an exact, analytical solution of the system of governing equations in terms of known functions at the present time. However, for the purpose of gaining some useful and reasonable understanding of the fracture problem defined by (10), (14), (15) and (20), let us replace the function $F[f,\sigma]$ by a function $\overline{F}(f)$ such that

$$\overline{F}(f) = \frac{1}{\tau}\int_0^{\tau} F[f,\sigma]dt \tag{21}$$

where integration is performed with respect to t by assuming f to be independent of time. Equation (21) can be obtained by minimizing the square error between the two functions over each cycle. For the class of problems under consideration here, this approximation is justified since for a short period τ the variation of f over any cycle can be assumed to be small. This may be expected from (14) and (15) since f changes from one to zero over a large number of cycles. Similar procedure based on (21) has been successfully used by Coleman and Marquardt[5] who obtained reasonable results for a certain range of loads and for large frequencies. Then using (21), we obtain a differential equation, equivalent to (10) in the form

$$\frac{df}{dt} = - \overline{F}(f) \tag{22}$$

which can be integrated by quadrature to give the time-to-break t_b as

$$t_b = \int_0^1 \frac{df}{\overline{F}(f)} \tag{23}$$

In (22) we have replaced (10) by averaging both sides over a cycle after assuming both f and df/dt to be constant during integration because of their slowly varying nature. If we consider that fracture does occur at the end of n_b cycles, then $t_b = \tau n_b$ and from (23)

$$n_b = \frac{1}{\tau}\int_0^1 \frac{df}{\overline{F}(f)} \tag{24}$$

Another way of approximating $F[f,\sigma]$ is to replace σ by a constant σ_o such that the impulses over half the cycle in both cases are equal. This yields

$$\sigma_o^{(1)} = \frac{2}{\tau}\int_0^{\tau/2} \sigma(\theta,\phi,t)dt \qquad (0 \le t < \frac{\tau}{2})$$

$$\sigma_o^{(2)} = \frac{2}{\tau}\int_{\tau/2}^{\tau} \sigma(\theta,\phi,t)dt \qquad (\frac{\tau}{2} \le t < \tau) \tag{25}$$

where $\sigma_o^{(1)}$ and $\sigma_o^{(2)}$ are functions of orientation (θ,ϕ) and the solution to (10) becomes

$$t_b = \int_0^1 \frac{df}{F[f,\sigma_o]} \tag{26}$$

which can be evaluated. This approximate solution (26) may be expected to be good for loads at high frequencies but it is difficult to estimate the errors involved. Also the computations are quite cumbersome since we must compute the value of f at the end of each half cycle in order to obtain t_b. This would involve solving a large number of algebraic equations with more and more terms as the frequency becomes larger and larger.

Having obtained the times-to-break $t_b(\theta,\phi)$ for individual representative elements, statistical methods can be used in obtaining the time-to-break $\overline{t}_b$ for the entire solid[3]. If $e_{mn}(t)$ are the small strains, then from the definition

$$\zeta(0)\,\mu(\theta,\phi) = Ee_{mn}(0)s_m s_n \quad \text{at } t = 0 \tag{27}$$

where E is the elastic constant for any representative element. Substituting into (1) we get

$$\sigma^{o}_{ij}\zeta(0) = EC_{ijmn}e_{mn}(0) \tag{28}$$

where

$$C_{ijmn} \cong \int \rho(\theta,\phi,\varepsilon)s_i s_j s_m s_n d\omega \tag{29}$$

If we define B_{ijmn} as the inverse of C_{ijmn} such that

$$\left\|B_{ijmn}\right\| = \left\|C_{ijmn}\right\|^{-1} \tag{30}$$

then

$$Ee_{mn}(0) = B_{ijmn}\zeta(0)\sigma^{o}_{ij}$$

and finally

$$\mu(\theta,\phi) = B_{ijmn}\sigma^{o}_{ij}s_m s_n \tag{31}$$

Here both C_{ijmn} and B_{ijmn} are functions of orientation strain ε. Substituting (31) into (23), we can evaluate the time-to-break $t_b(\theta,\phi)$ for a representative element directed along (θ,ϕ). Equation (31) defines a generalized surface for each direction (θ,ϕ) in six coordinates σ^{o}_{ij} and determines whether a given element will ever break or not under a given state of loading. The statistical value $\overline{t}_b$ for a complete solid system is:

$$\bar{t}_b = \frac{\int t_b(\theta,\phi,\varepsilon)\ \rho(\theta,\phi,\varepsilon)d\omega}{\int \rho(\theta,\phi,\varepsilon)d\omega} \tag{32}$$

provided the integrals exist. Here we have assumed a continuous distribution of elements throughout the solid. However, no difficulties are encountered in case the distribution is discrete. Through the use of the relation

$$\int \rho(\theta,\phi,\varepsilon)d\omega = 1$$

we can rewrite (32) as

$$\bar{t}_b = \int t_b(\theta,\phi,\varepsilon)\ \rho(\theta,\phi,\varepsilon)d\omega \tag{33}$$

which can be easily computed once t_b is known.

For a completely oriented solid, in which all the elements are directed along x_3 - direction, subjected to a uniform simple tension $\sigma_{33}(t) = \sigma^{o}_{33}\zeta(t)$, equations (9) and (10) reduce to

$$f(t)\ \psi(t) = \sigma^{o}_{33}\zeta(t) \tag{34}$$

and

$$\frac{df}{dt} = -\ F[f,\sigma^{o}_{33}\zeta(t)] \tag{35}$$

which can be easily solved. All the preceding equations for a general partially oriented solid apply to completely oriented solid if we substitute $\mu(\theta,\phi)$ by σ^{o}_{33}. The entire solid will break at the same time as any representative individual element since the solid is assumed to be homogeneous with perfectly oriented elements.

In order to obtain some idea about the results, let us specialize to the following types of loading.

1) Rectangular Pulses

When the periodic loading history is a series of rectangular pulses defined as

$$\sigma_{33}^{o}\zeta(t) = \begin{cases} \sigma_1 & (0 \le t < x\tau) \\ \sigma_2 & (x\tau \le t < \tau) \end{cases} \tag{36}$$

where $0 < x \le 1$, then $\overline{F}(f)$ becomes

$$\overline{F}(f) = xF[f,\sigma_1] + (1 - x)\, F[f,\sigma_2] \tag{37}$$

Substituting into (21) we can compute t_b. Some numerical results are obtained for $x = 1/2$ and assumed constants γ, β, ω_r and ω_b (for aluminum at 400° C)[4] and various values of $\xi = \sigma_2/\sigma_1$. The results are shown in Figure 1.

2) Saw Tooth Loads

If the loading function is a series of triangular pulses given by

$$\sigma_{33}^{o}\zeta(t) = \begin{cases} \sigma_2 + \dfrac{\sigma_1 - \sigma_2}{x\tau}\, t & (0 \le t \le x\tau) \\ \sigma_1 - \dfrac{\sigma_1 - \sigma_2}{(1-x)\tau}(t - x\tau) & (x\tau \le t \le \tau; \quad x < 1) \end{cases} \tag{38}$$

then $\overline{F}(f)$ becomes

$$\overline{F}(f) = \frac{1}{\sigma_1 - \sigma_2} \left\{ \frac{\Omega_b}{\beta} f^2[\exp(\frac{\beta\sigma_1}{f}) - \exp(\frac{\beta\sigma_2}{f})] \right.$$
$$\left. + \frac{\Omega_r}{\gamma} f(1 - f)[\exp(- \frac{\gamma\sigma_1}{f}) - \exp(- \frac{\gamma\sigma_2}{f})] \right\} \qquad (\ldots)$$

If, instead of (38) we have a loading

$$\sigma_{33}^{o}\zeta(t) = \begin{cases} \sigma_2 + \dfrac{\sigma_1 - \sigma_2}{\tau} t & (0 \leq t < \tau) \\ \sigma_2 & (t = \tau) \end{cases}$$

We again obtain the same expression as in (39) for $\overline{F}(f)$. It is clear from (39) that the function $\overline{F}(f)$ is independent of x. Numerical results in this case are shown in Fig. 2 for various values of $\xi = \sigma_2/\sigma_1$ assuming the same values for constants γ, β, ω_r and ω_b as in the previous case.

3) Sinusoidal Loads

The sinusoidal loading history of the following form is of importance because of obvious practical engineering applications.

$$\sigma_{33}^{o}\zeta(t) = \sigma_o + \sigma_1 \sin(2\pi pt) \qquad (\sigma_o \geq \sigma_1 > 0) \qquad (40)$$

where σ_o is the static mean or average stress. A large number of experimental results deal with this type of loading history. Consequently, there exists a good deal of experimental data for comparison. In this case we obtain

$$\overline{F}(f) = \Omega_b f \exp\left(\frac{\beta\sigma_o}{f}\right) I_o\left(\frac{\beta\sigma_1}{f}\right) - \Omega_r(1-f) \exp\left(-\frac{\gamma\sigma_o}{f}\right) I_o\left(\frac{\gamma\sigma_1}{f}\right) \quad (41)$$

where $I_o(x) = I_o(-x)$ is the zeroth order hyperbolic Bessel function. Through the use of (41), we can compute time-to-break t_b easily for various values of σ_o and σ_1. Unfortunately the integral cannot be expressed in a closed form. Numerical results based on constants, assumed to be same as previous cases are shown in Figure 3.

It should be pointed out that all of the numerical results are motivated toward establishing the qualitative nature of the time dependent fracture behavior only. It appears that it is successful as far as this purpose is concerned. However, for quantitative comparison with any experimental data, the various constants in the formulation will have to be properly determined.

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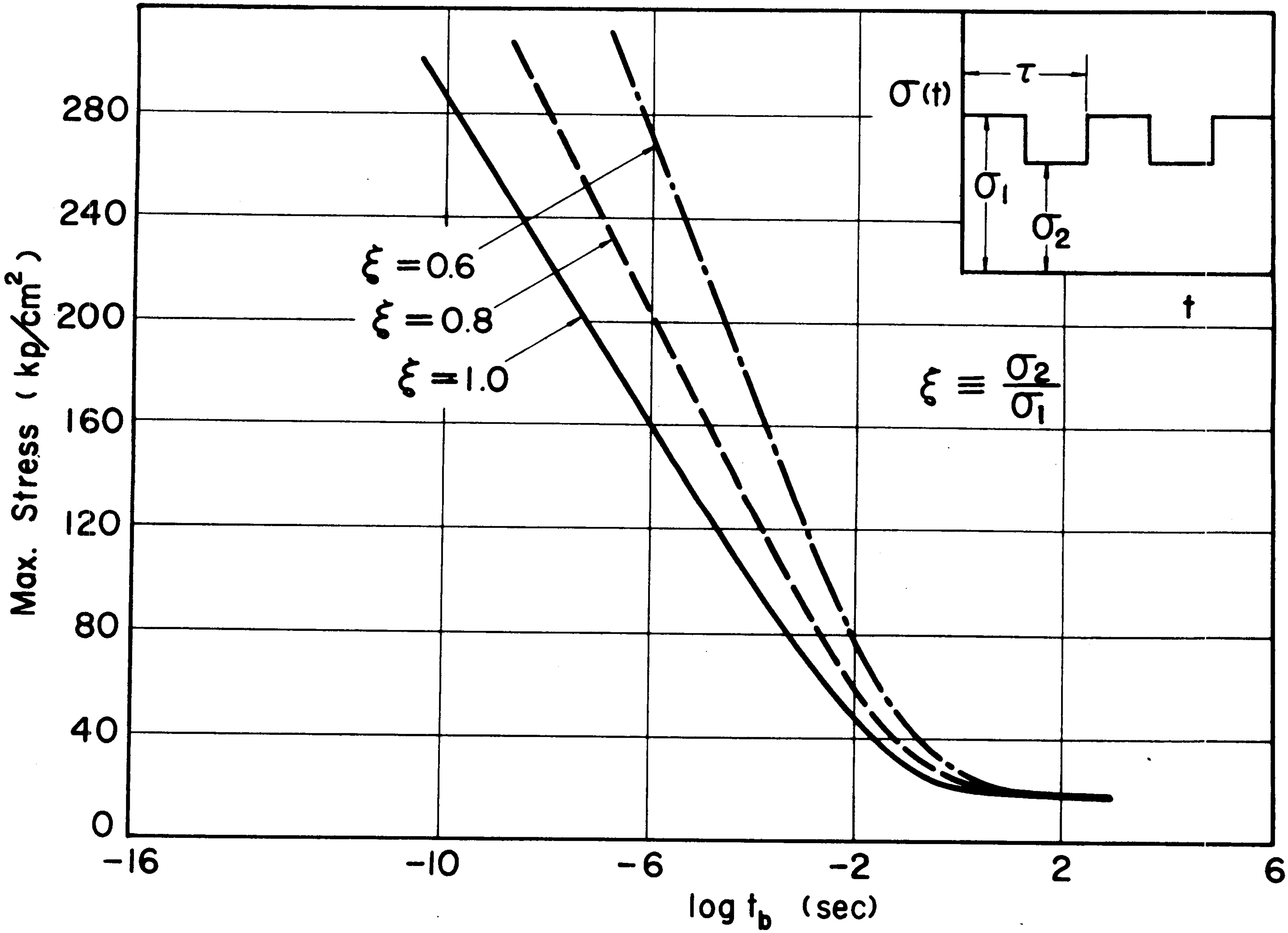

Fig. 1 Fracture Time for Rectangular Wave Loading

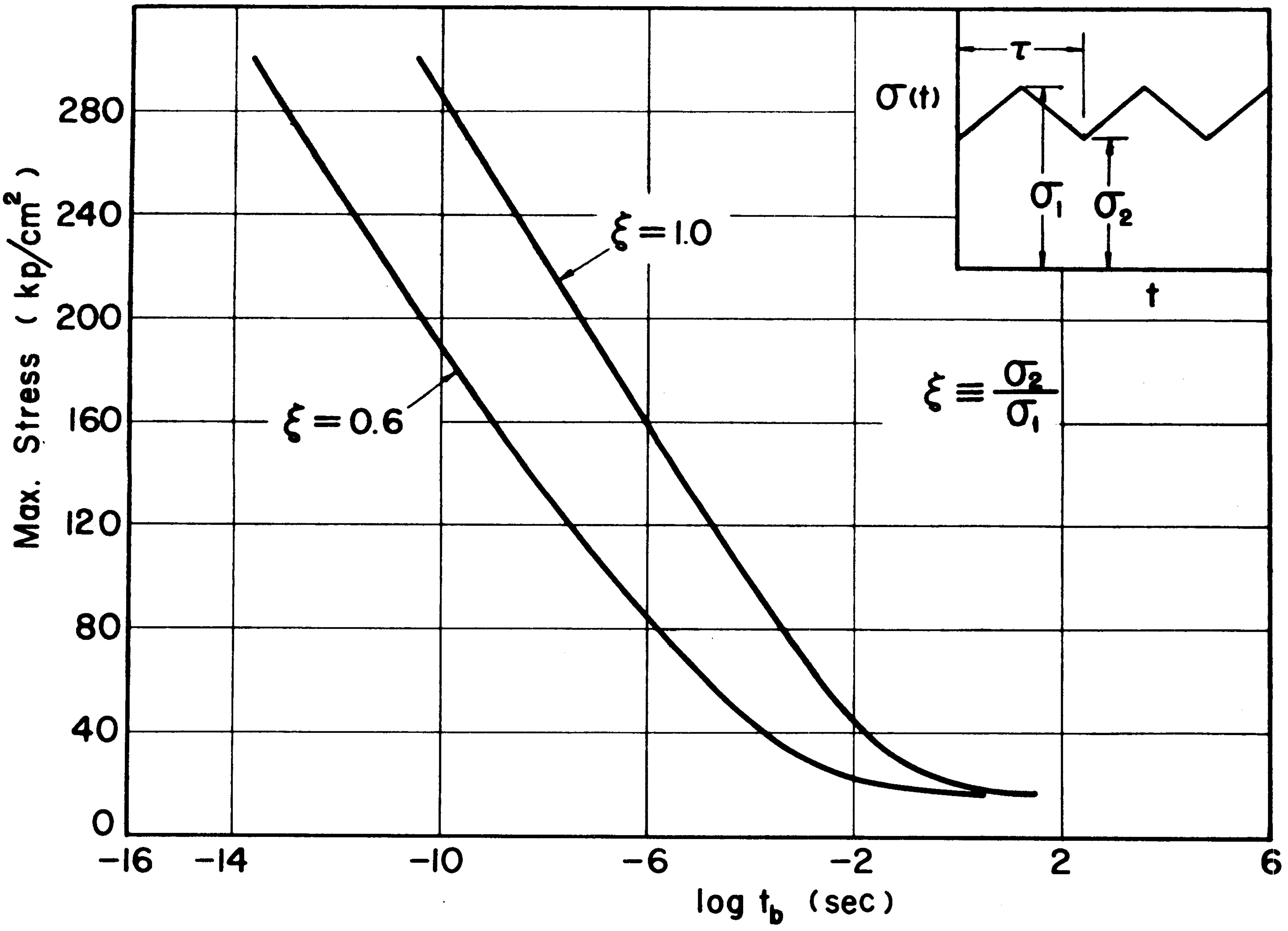

Fig. 2 Fracture Time For Triangular Wave Loading

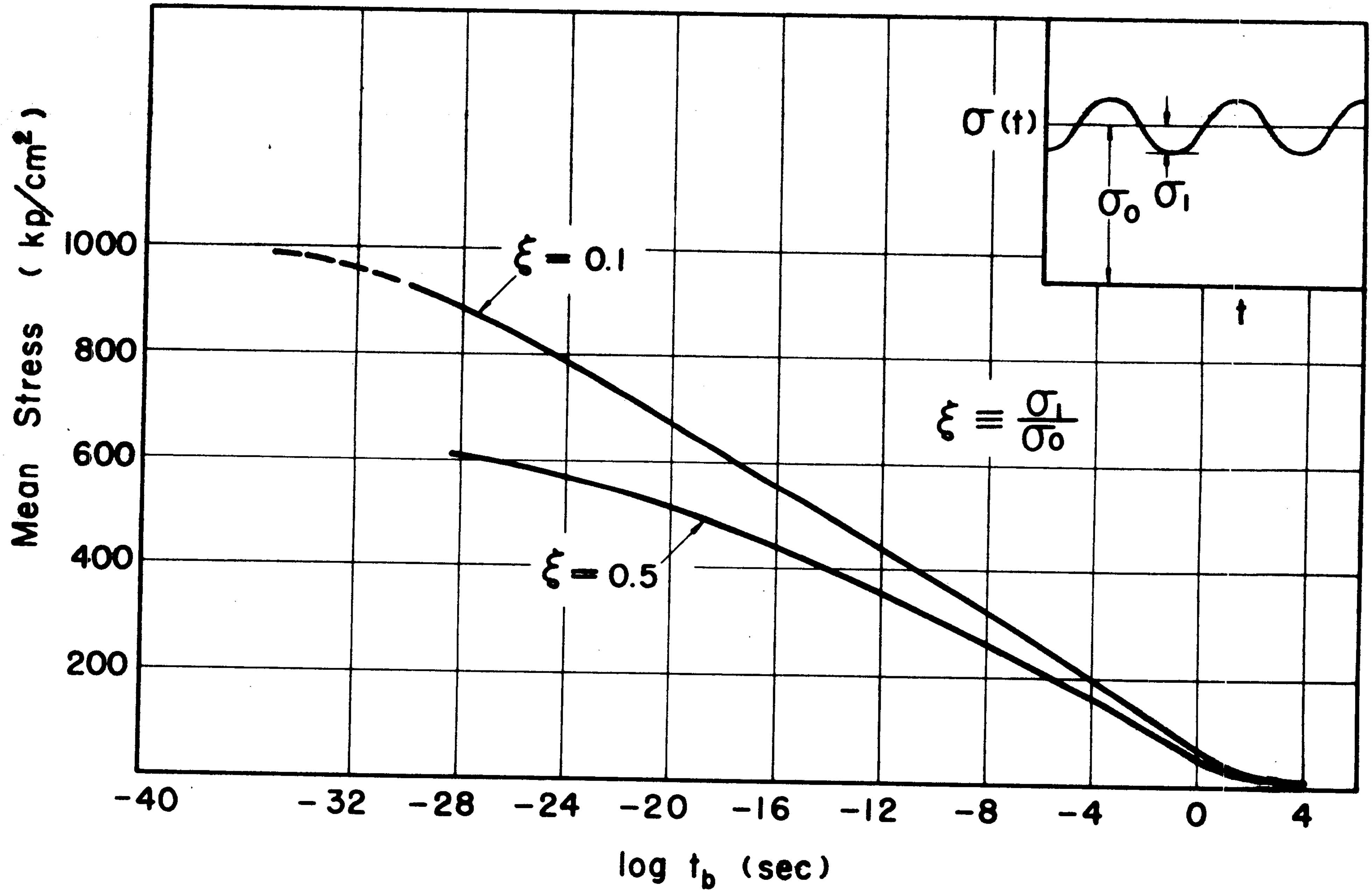

Fig. 3 Fracture Time for Sinusoidal Wave Loading